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HOROCYCLE FLOWS ARE LOOSELY BERNOULLI

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ABSTRACT
It is proved that horocycle flows associated with transitive C^2-Anosov flows are loosely Bernoulli with respect to their unique ergodic measures.

Let T_t be an Anosov flow of class C^2 on a compact smooth n-dimensional Riemann manifold M and W^s, W^u, W^{ss}, W^{uu} be the stable, unstable, strong stable, strong unstable foliations of T_t (see [1,5]). Suppose that W^{uu} is orientable and one-dimensional. Then the unit tangent bundle of W^{uu} defines a continuous flow H_t on M which is called the horocycle flow associated with T_t. The orbits of H_t are precisely the leaves of W^{uu}.

When T_t is the geodesic flow on the space M of unit tangent vectors of a compact surface of constant negative curvature then H_t is the well known classical horocycle flow ([2], [10], [11], [20]). Both T_t and H_t preserve the Riemannian volume on M, are ergodic,

(1) $$T_t \cdot H_s = H_{\lambda^t s} \cdot T_t \quad \text{for all } s, t \in R \text{ and some } \lambda > 1,$$

and every orbit of H_t is dense in M. H. Furstenberg [9] has proved that H_t is uniquely ergodic.

It turned out that similar properties are enjoyed by any transitive (i.e., leaves of $W^{uu}W^{ss}$ are dense in M) Anosov T_t of class C^2. More precisely, it has been proved in [6], [16], [17], [19] that in this case there is a continuous reparametrization ϕ: $M \times R \to R$†† on orbits of H_t s.t. ϕ preserves orientation and the flow

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††φ might be not absolutely continuous with respect to t. In this case it is not a legitimate time-change on orbits of H_t.
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h_t: $h_t x = H_{\phi(x,t)} x$ satisfies the uniform expanding condition (1). R. Bowen and B. Marcus [6], [17] have proved that h_t (and therefore H_t since ϕ is continuous) is uniquely ergodic. The unique ergodic measure μ of h_t is positive on open sets, and is also preserved by T_t. Actually, it is the measure of maximal entropy for T_t ([6], [17]). B. Marcus [18] proved that (h_t, μ) is mixing of all degrees (see [28]).

h_t and H_t have zero entropy.† This property is common for all measurable flows having the same orbits and time-orientation. In other words, zero entropy is invariant under Kakutani equivalence (see [8], [12], [13], [21], [30]).

Recently, J. Feldman [8] and A. B. Katok [13] have introduced independently a new property, loosely Bernoulli (LB) by J. Feldman, which is also invariant under Kakutani equivalence. A zero entropy ergodic flow S_t is LB if after a suitable measurable time-change it becomes isomorphic to an irrational flow on 2-torus. Equivalently, S_t is LB iff it is isomorphic to a special flow built over an irrational rotation of the circle with a positive integrable function f. It follows from [7], [23] that f can be taken differentiable on the circle, but the derivative of f might be unbounded.

We prove the following

THEOREM 1. *Let T_t be a transitive Anosov flow of class C^2 on a compact M with W^{uu} orientable and one-dimensional. Then the associated horocycle flow H_t is LB with respect to its unique ergodic measure.*

COROLLARY 1. *Let T_t be the geodesic flow of the unit tangent bundle of a compact manifold of negative curvature with W^{uu} as in Theorem 1. Then the associated horocycle flow is LB with respect to its unique ergodic measure.*

A Borel probability measure μ on M is called natural if $d\mu = d\nu \times dt$ where dt is the Lebesque measure on orbits of h_t and ν is a measure on leaves of W^s s.t. ν is positive on open sets and for each $x \in M$, $r > 0$ there is a compact $A \ni x$, $A = \overline{\mathrm{Int}\, A} \subset W^s(x)$, $\mathrm{diam}\, A < r$ s.t. the ν-measure of the boundary ∂A of A is zero.

It follows from [6] that the unique ergodic measure of h_t mentioned above is natural.

One can see that actually our proof of Theorem 1 works for a more general Theorem where compactness M is not assumed.

THEOREM 2. *Let M be a countable union of compact sets and T_t be an Anosov*

†This follows from the LB-property we prove below.

flow on M. Let W^{uu} be orientable and one-dimensional. Let h_t be a continuous flow on M whose orbits are leaves of W^{uu}. Assume that (1) *$T_t \cdot h_s = h_{\lambda^t s} \cdot T_t$ for all $s, t \in R$ and some $\lambda > 1$;* (2) *both T_t and h_t preserve a natural measure μ on M;* (3) *(h_t, μ) is ergodic. Then (h_t, μ) is LB.*

Theorem 2 implies

THEOREM 3. *Let T_t be the geodesic flow on the unit tangent bundle M of a manifold X of constant negative curvature. Let* vol $X < \infty$. *Then the horocycle flow H_t on M is LB with respect to the invariant Riemannian volume* (vol) *on M.*

We remark that M in the Theorem 3 is not necessarily compact. It has been proved in [2], [11], [20] that if vol $X < \infty$ both (T_t, vol) and (H_t, vol) are ergodic.

(1) shows that h_t possesses the following property:

(2) For any $u, v > 0$, h_u and h_v are isomorphic.

It follows from our Theorems that

COROLLARY 2. *Any LB ergodic flow with zero entropy can be time-changed to possess* (2), *to become mixing of all degrees* [18] *and to have denumerably infinite Lebesgue spectrum* [24].

QUESTION. Can any zero entropy ergodic flow be time changed to possess (2), to become mixing of all degrees, to have denumerably infinite Lebesgue spectrum?

We mention that D. Ornstein and M. Smorodinsky [22] have proved that any positive entropy ergodic flow can be time changed to become a K-flow.

B. Weiss [30] proved that a zero-entropy ergodic flow S_t is LB iff S_{t_0} is LB for at least one particular t_0 (then for all t) (see D. Rudolph [26] for positive entropy case). Therefore h_1 is LB.

QUESTION. Are $h_t \times h_t$ and $h_1 \times h_1$ LB?

We remark that Kushnirenko [15] showed for constant negative curvature case that $h_1 \times h_1$ is not isomorphic to h_1 while D. Ornstein and D. Rudolph [27] gave an example of an LB T s.t. $T \times T$ is not LB.

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1. The $\bar{f}$-metric (see [21], [30])

We are going to prove that (h_t, μ) is LB, where h_t satisfies (1) and $d\mu = d\nu \times dt$ is h_t-invariant natural measure. Sometimes for simplicity we will use the unique ergodicity of h_t; it is not essential and enough to have just ergodicity (we care about Theorems 2, 3 where h_t is not necessarily uniquely ergodic).

Let $w, w' \in \{1, 2, \cdots, a\}^n$. Then $\bar{f}_n(w, w') = 1 - k/n$ where k is the maximal integer for which we can find subsequences $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_k$ with $w(i_r) = w'(j_r)$, $1 \leqq r \leqq k$.

Let T be a zero-entropy m.p.t. in a probability space (X, μ) and let $\boldsymbol{P} = \{P_1, \cdots, P_a\}$ be a partition of X. If $x \in P_j$ then j is the $\boldsymbol{P}$-name of $x \in X$. Denote $w_n(x) = \{x_1, \cdots, x_n\}$ where x_i is the T^{-i} $\boldsymbol{P}$-name of x.

$\boldsymbol{P}$ is called an LB-partition if given $\varepsilon > 0$ there is $N > 0$ and a set $Y \subset X$, $\mu(Y) > 1 - \varepsilon$ s.t. if $n \geqq N$ and $x, y \in Y$ then $\bar{f}_n(w_n(x), w_n(y)) < \varepsilon$. We say that $w_n(x)$ and $w_n(y)$ or $\{x, Tx, \cdots, T^n x\}$ and $\{y, Ty, \cdots, T^n y\}$ are ε-$\boldsymbol{P}$-matchable.

An ergodic T is LB if it has an LB generator.

2. u-cylindric partitions

Let W be a foliation in M and $A \subset M$. We write $A \subset W$ if A is a subset of a leaf of W.

Definition 1. Two sets $A, B \subset W^s$ are called u-isomorphic ($A \overset{u}{\sim} B$) compare with the canonical isomorphism in [3], [4], [25], [29]) if there is a continuous $g: A \times I \to M$ ($I = [0, 1]$) s.t. (1) $g(x, I) \subset W^{uu}$, $x \in A$, (2) $g(x, 0) = x$ $g(x, 1) \in B$ and the map $\tilde{g}: A \to B$ $\tilde{g}(x) = g(x, 1)$ is a homeomorphism. The set $g(A \times I) = P$ is called a u-cylinder with faces A, B. If the positive direction on orbits of h_t goes from A to B we write $A = A_1(P)$ and $B = A_2(P)$. $g(x, I)$ and $g(y, I)$, $x, y \in A$ are called s-isomorphic ($g(x, I) \overset{s}{\sim} g(y, I)$).

We denote η_P the partition of P into sets u-isomorphic to $A_1 = A_1(P)$ and $\xi_P = \{g(x, I) \subset P, x \in A_1\}$ the partition of P into s-isomorphic intervals of orbits of h_t.

Henceforth we suppose that $A_1 = \overline{\operatorname{Int} A_1} \subset W^s$ is compact and the ν-measure of the boundary ∂A_1 of A_1 is zero.

Let $\alpha = \{P_1, \cdots, P_m\}$ be a partition of M into u-cylinders (we say α is a u-partition).

We may get such an α by the following procedure (see [3], [4], [25], [29]). Let

$\sigma = \{Q_1, \cdots, Q_q\}$ be a cover of M by small u-cylinders. Suppose that $\operatorname{Ind} Q_i \cap \operatorname{Int} Q_j \neq \emptyset$, $i \neq j$. Then there are $A_i \in \eta_{Q_i}$, $A_j \in \eta_{Q_j}$ s.t. $B_{ij} = A_i \cap A_j = \overline{\operatorname{Int}(A_i \cap A_j)} \subset W^s$ has non-empty interior. Using the u-isomorphism we may partition $Q_i \cup Q_j$ into finite many u-cylinders whose faces are u-isomorphic images of B_{ij} and of $\overline{A_i - B_{ij}}$, $\overline{A_j - B_{ij}}$. The boundaries of the faces are contained in a union of some u-isomorphic images of $\partial A_1(Q_i)$ and $\partial A_1(Q_j)$. ν is zero on $\partial A_1(Q_i)$ and $\partial A_1(Q_j)$. Since μ is h_t-invariant and $d\mu = d\nu \times dt$ ν is invariant under the u-isomorphism. Therefore the ν-measure of the boundaries is 0. We get a finite u-partition α applying this process inductively to the cover σ.

Denote $\xi_\alpha = \{C \in \xi_P, P \in \alpha\}$ and $\eta_\alpha = \{A \in \eta_P, P \in \alpha\}$.

Let $\beta = \{A_1(P): P \in \alpha\}$ and $X = \bigcup \beta$ be the set-theoretic union of atoms of β. For $x \in X$ we denote $\psi(x)$ the first intersection of the orbit $\{h_t(x), t > 0\}$ with X and $F(x)$ the length of $[x, \psi(x)]$ on the orbit. Actually $[x, \psi(x)] \in \xi_\alpha$. (h_t, μ) is a special flow (ψ, F) built over (X, ψ, ν) with F, where ν is a ψ-invariant ergodic Borel measure on X s.t. $d\mu = d\nu \times dt/\bar{F}$ where $\bar{F} = \int F d\nu$. We say that (X, ψ, ν) is a cross-section of (h_t, μ). By Abramov formula ψ has zero entropy. (h_t, μ) is LB iff (X, ψ, ν) is LB (see [8], [13], [30]).

We will prove that β is LB for (ψ, ν).

For $u \in M$ we denote by $I_t(u)$ the orbit interval $[u, h_t u]$, $t > 0$. Since α is a u-partition, $I_t(u)$ can be uniquely represented as a disjoint union of intervals $\bigcup_{i=0}^{i(u)} J_i(u)$, $J_i < J_{i+1}$, where $J_0(u)$, $J_{i(u)}(u) \subset C \in \xi_\alpha$ and $J_i(u) \in \xi_\alpha$ for $i = 1, \cdots, i(u)-1$. If $J_i(u) \subset \xi_{P_m}$ for some $P_m \in \alpha$ we say that m is the α-name of $J_i(u)$. If $J_i(u) = [u_i, u_{i+1}]$ then $u_i \in X$ for $i = 1, \cdots, i(u)$.

Definition 2. For $u, v \in M$, $I_t(u)$ and $I_t(v)$ are called ε-α-matchable if there are subsequences $1 \leqq i_1 < i_2 < \cdots < i_k \leqq i(u)-1$, $1 \leqq j_1 < j_2 < \cdots < j_k \leqq i(v)-1$ s.t. $J_{i_p}(u)$ and $J_{j_p}(v)$ have equal α-names, $p = \overline{1, k}$, and the measures $l(\bigcup_{p=1}^k J_{i_p}(u))/t$, $l(\bigcup_{p=1}^k J_{j_p}(v))/t$ are at least $1-\varepsilon$.

Lemma 1. *Suppose that given $\delta > 0$ there is $t_0 = t_0(\delta)$ s.t. if $t \geqq t_0$ then for any $u, v \in M$ $I_t(u)$ and $I_t(v)$ are δ-α-matchable. Then the partition β is LB for (X, ψ, ν).*

Proof. We consider the special flow $(h_t, \mu) = (\psi, F, \nu)$. Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ be chosen later. Since ψ is ergodic there is $n_0 = n_0(\delta)$ and a set $X_0 \subset X$, $\nu(X_0) > 1 - \delta$ s.t. if $x \in X_0$, $n \geqq n_0$ then

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} F(\psi^i x) - \bar{F} \right| < \delta \qquad \text{or}$$

$$\left| \sum_{i=0}^{n-1} F(\psi^i x) - n\bar{F} \right| < n\delta. \tag{3}$$

Let $N_0 = N_0(\delta) = \max\{n_0(\delta), t_0(\delta)/\bar{F}\}$. We claim that if $n > N_0$ then $w_n(x)$ and $w_n(y)$ are $K\delta$-β-matchable for any $x, y \in X_0$, where $K = (3+2\bar{F})/a$ and $a = \inf_X F(x)$.

Consider the orbit intervals $[x, \psi^n x]$ and $[y, \psi^n y]$. $l[x, \psi^n x] = \sum_{i=0}^{n-1} F(\psi^i x)$ and $l[y, \psi^n y] = \sum_{i=0}^{n-1} F(\psi^i y)$. Let $t = n\bar{F} > t_0(\delta)$. By assumption $I_t(x)$ and $I_t(y)$ are δ-α-matchable. By (3) we have

$$|t - l[x, \psi^n x]| < n\delta, \qquad |t - l[y, \psi^n y]| < n\delta. \tag{4}$$

Let $I_t(x) = \bigcup_{i=0}^{i(x)} J_i(x)$, $J_i(x) = [x_{i-1}, x_i]$, $x_0 = x$. All points $x_i \in X$, $i = 0, \cdots, i(x) - 1$ and by (4)

$$|n - i(x)| < n\delta/a, \qquad |n - i(y)| < n\delta/a. \tag{5}$$

Let $\alpha_t(x) = \{J_i(x)\colon i = 0, \cdots, i(x)\}$ and $\gamma_t(x) = \{J_{i_p}(x) \in \alpha_t(x)\colon J_{i_p}(x)$ is δ-α-matched with some $J_{j_p}(y) \in \alpha_t(y)$ in a δ-α-match between $I_t(x)$ and $I_t(y)\}$. So $J_{i_p}(x) = [x_{i_p-1}, x_{i_p}]$ and $J_{j_p}(y) = [y_{j_p-1}, y_{j_p}]$ have equal α-names and then x_{i_p-1} and y_{j_p-1} have equal β-names. So we match x_{i_p-1} with y_{j_p-1} in the sequences $\{x_0 = x, x_1, \cdots, x_{i(x)}\}$ and $\{y_0 = y, y_1, \cdots, y_{i(y)}\}$ where in general $i(x) \neq i(y)$.

The total length of intervals of $\alpha_t(x)$ which are not in $\gamma_t(x)$ is at most $t\delta$. So the number of points in $\{x, x_1, \cdots, x_{i(x)}\}$ which are not matched is at most $2t\delta/a = 2n\bar{F}\delta/a$.

Let $m = \min\{i(x), i(y), n\}$. By (5) we have

$$|m - n| < n\delta/a, \qquad |m - i(x)| < 2n\delta/a, \qquad |m - i(y)| < 2n\delta/a. \tag{6}$$

So at least $m - 2t\delta/a$ points of $\{x, x_1, \cdots, x_m\}$ are matched. The same way at least $m - 2t\delta/a$ points of $\{y, y_1, \cdots, y_m\}$ are matched. By (6) at least $m - 2t\delta/a - 2n\delta/a$ points of $\{x, \cdots, x_m\}$ are matched with some points of $\{y, \cdots, y_m\}$.

By (6) $m > n - n\delta/a$ so at least $n - n\delta/a - 2n\bar{F}\delta/a - 2n\delta/a$ points of $\{x, x_1, \cdots, x_n\}$ are δ-matched with some points of $\{y, y_1, \cdots, y_n\}$. This says that $\bar{f}_n(w_n(x), w_n(y)) < K\delta$ where $K = (3+2\bar{F})/a$. So given $\varepsilon > 0$ we chose $\delta > 0$ s.t. $K\delta < \varepsilon$. Then for $n > N_0(\delta)$, $\bar{f}_n(w_n(x), w_n(y)) < \varepsilon$, $x, y \in X_0(\delta)$. ■

3. We are going to prove that h_t satisfies the condition of Lemma 1.

For a u-cylinder P let $O_\gamma(A_1(P))$ be the γ-neighborhood of the boundary $\partial A_1(P)$ in $W^s(A_1(P))$. Let $Q = Q_\gamma(P)$ be the u-cylinder with $A_1(Q) = O_\gamma(A_1(P))$ and $A_2(Q) \subset W^s(A_2(P))$. Let $\Delta_\gamma(\alpha) = \bigcup_{P\in\alpha} Q_\gamma(P)$. Let $\delta > 0$ be fixed and $\gamma > 0$ be so small that $\mu(\Delta_\gamma(\alpha)) < \delta/2$.

We use unique ergodicity of h_t to get $N = N(\delta) > 0$ s.t.

$$\max_{C\in\xi_\alpha} l(C)/N < \delta \qquad \text{and}$$

(7) if $t > N$ and $x \in M$ then the relative Lebesgue measure of the set $\Delta_\gamma(\alpha)$ on $I_t(x)$ is at most δ.

Let $\gamma' = \inf\{d_s(x, y)\colon x \in \partial A,\ y \in A \cap \partial P,\ A \in \eta_{Q_\gamma(P)},\ P \in \alpha\} > 0$ where d_s is the metric on leaves of W^s.

Let $0 < \delta' < \gamma'/2$ be s.t. if $A \subset W^s$, $\operatorname{diam}_s A < \delta'$, $x \in A$ and P is a u-cylinder with $A_1(P) = A$ and $[x, h_N(x)] \in \xi_P$ then

$$\operatorname{diam}_s B < \gamma'/2 \text{ for any } B \in \eta_P,$$

(8) $$|l(C') - l(C'')| < \delta \quad \text{for any } C', C'' \subset P,\ C' \overset{s}{\sim} C'' \qquad \text{and in particular}$$

$$|l(C) - N| < \delta \quad \text{for any } C \in \xi_P.$$

Let $t \geqq N$ and $s = s(t)$ be s.t. $\lambda^s \cdot N = t$ (see (1)).

The following lemma contains the main point of the proof.

Lemma 2 (Basic). *Let $\delta > 0$ be given and let $N = N(\delta)$ and $\delta' = \delta'(\delta)$ be as above. Let $B \subset M$ and* diam $B < \delta'$. *Then for any $u, v \in B$ and any $t \geqq N$, $I_t(T_s u)$ and $I_t(T_s v)$ are 10δ-α-matchable.*

Proof. We consider the orbit intervals $I_N(u)$ and $I_N(v)$. Let $I(u)$, $I(v)$ be the maximal s-isomorphic subintervals of $I_N(u)$, $I_N(v)$. It follows from (8) that

(9) $$|l(I(u)) - N| < 4\delta, \qquad |l(I(v)) - N| < 4\delta.$$

Let $u_s = T_s u$ and $I(u_s) = T_s I(u)$. $I(u_s)$ and $I(v_s)$ are s-isomorphic and it follows from (1) and (9) that

(10) $$|l(I(u_s)) - t| < 4\delta \cdot t, \qquad |l(I(v_s)) - t| < 4\delta \cdot t.$$

Let $I(u_s) = \bigcup_{i=0}^{p} J_i(u_s)$, $I(v_s) = \bigcup_{j=0}^{q} J_j(v_s)$ be the above decompositions into α-named intervals.

Let some $[x', x''] \subset I(u_s)$ be s-isomorphic to some $[y', y''] \subset I(v_s)$. x', y' are in the same leaf of W^s and so are x'', y''. Since T contracts W^s it follows from (8) that $d_s(x', y')$, $d_s(x'', y'') < \gamma'/2$. By definition of γ' we have

(11) x' and y' have different α-names only if $x', y' \in \Delta_\gamma(\alpha)$.

Let $J_i(u_s) = [x'_i, x''_i]$, $J_i(v_s) = [y'_j, y''_j]$ and let

$$\tilde{J}(u_s) = \{J_i(u_s): x'_i \notin \Delta_\gamma(\alpha) \text{ or } y(x'_i) \notin \Delta_\gamma(\alpha), [x'_0, x'_i] \overset{s}{\sim} [y'_0, y(x'_i)]\},$$

$$\tilde{J}(v_s) = \{J_j(v_s): y'_j \notin \Delta_\gamma(\alpha) \text{ or } x(y'_j) \notin \Delta_\gamma(\alpha), [y'_0, y'_j] \overset{s}{\sim} [x_0, x(y'_j)]\}.$$

By (7) and (10) the total relative Lebesgue measure of $\bigcup \tilde{J}(u_s)$ and of $\bigcup \tilde{J}(v_s)$ on $I_t(u_s)$ and $I_t(v_s)$ is at least $1-5\delta$.

It follows from (11) that $\tilde{J}(u_s)$ and $\tilde{J}(v_s)$ have the same number of intervals and if we label them in the increasing order

$$\tilde{J}(u_s) = \{F_1(u_s), \cdots, F_r(u_s)\}, \qquad \tilde{J}(v_s) = \{F_1(v_s), \cdots, F_r(v_s)\}$$

then $F_i(u_s)$ and $F_i(v_s)$ will have equal α-names. This completes the proof. ■

Let $\varepsilon > 0$ be fixed and $\delta = \delta(\varepsilon) = \varepsilon/10$. Let $\omega = \omega(\delta) = \{B_1, \cdots, B_r\}$ be a finite partition of M into sets of positive μ-measure with $\operatorname{diam} B_i < \delta'(\delta)$, $i = \overline{1, r}$. We get from Lemma 2

Corollary. *Given $\varepsilon > 0$ there is $N = N(\varepsilon) > 0$ s.t. for any $t \geqq N$ there is a partition $\omega_t = \omega_t(\varepsilon) = \{B'_1, \cdots, B'_r\}$, $B'_i = T_s B_i$, $B_i \in \omega(\delta)$, $i = \overline{1, r}$ s.t. if $u, v \in B'_i$ then $I_t(u)$ and $I_t(v)$ are ε-α-matchable. Since T_t preserves μ, $\mu(B'_i) = \mu(B_i)$ for all $t \geqq N$, $i = \overline{1, r}$.*

The following definition is quite analogical to B. Weiss' (ε, M, c)-matchability [30].

Definition 3. $I_t(u)$ and $I_t(v)$ are called $(\theta, \varepsilon, M, c)$-$\alpha$-matchable if there are disjoint subintervals

$$S_1(u) < S_2(u) < \cdots < S_k(u), \qquad S_i(u) \subset I_t(u), \quad i = \overline{1, k},$$

$$S_1(v) < S_2(v) < \cdots < S_k(v), \qquad S_i(v) \subset I_t(v),$$

and a subset $J \subset \{1, 2, \cdots, k\}$ s.t.

(1) $l(\bigcup_{i=1}^k S_i(u))/t$, $l(\bigcup_{i=1}^k S_i(v))/t > 1 - \theta$,
(2) $l(\bigcup_{i \in J} S_i(u))/t$, $l(\bigcup_{i \in J} S_i(v))/t \geqq c$,
(3) if $i \in J$ then $S_i(u)$ and $S_i(v)$ are ε-α-matchable,
(4) if $i \notin J$ then $l(S_i(u)), l(S_i(v)) \geqq M$.

We repeat B. Weiss' proposition 7.5 in [30] (see also proposition 1 in [14] by Katok and Sataev) to get

LEMMA 3. *Let $\varepsilon > 0$ be given and $c = c(\varepsilon) = \min\{\mu(B_i), B_i \in \omega(\varepsilon)\} > 0$. For any $\theta > 0$ and $M > 0$ there is $n_0 = n_0(\varepsilon, M, \theta)$ s.t. if $t \geqq n_0$ then for any $u, v \in M$ $I_t(u)$ and $I_t(v)$ are $(\theta, \varepsilon, M, c/2)$-$\alpha$-matchable.*

PROOF. Take $\delta = \delta(\varepsilon) = \varepsilon/L$ and $N = N(\delta)$. Let $t_0 > \max\{N, M\}$ and $\omega_{t_0} = \{B_1^{t_0}, \cdots, B_r^{t_0}\}$. $\mu(B_i^{t_0}) \geqq c > 0$, $i = \overline{1, r}$. Denote $h = h_1$. We use the unique ergodicity of h to get $n > 0$ s.t. if $x \in M$ then the frequency of each $B_i^{t_0}$, $i = \overline{1, r}$ in the sequence $\{x, hx, \cdots, h^n x\}$ is at least $3c/4$. Let $n_0 > t_0/\theta$ be so big that $n/n_0 < \theta/2$. Let $t \geqq n_0$. Take $I_t(u)$ and $I_t(v)$. $I_t(u) = I_{t_0}(u) \cup I_{t_0}(u_{t_0}) \cup \cdots = S_1(u) \cup S_2(u) \cup \cdots \cup S_p(u)$ where $l(S_i) = t_0$, $i = 1, 2, \cdots, p-1$, $l(S_p(u)) \leqq t_0$. Write underneath $I_t(u)$ the shifts $I_t(v), hI_t(v), h^2 I_t(v), \cdots, h^n I_t(v)$. Under every point of the form u_{it_0}, $i = 0, 1, 2, \cdots, p-1$ are written the points v_{it_0}; $hv_{it_0}, \cdots, h^n v_{it_0}$. By our choice of n in each such column the frequency of sets $B_i^{t_0}$, $i = \overline{1, r}$ is at least $3c/4$. If a point w of the column belongs to $B^{t_0}(u_{it_0})$ we call such an occurrence a desirable event, $i = \overline{1, p}$. In this case $I_{t_0}(u_{it_0})$ and $I_{t_0}(w)$ are ε-α-matchable.

The frequency of the desirable event in each column is at least $3c/4$ and using simple arguments (see for instance lemma 1 in [14]) we conclude that there is $0 \leqq s \leqq n$ s.t. on the shift $h^s I_t(v)$ the frequency of desirable intervals of length t_0 is at least $3c/4$. Since $t_0 > M$ this obviously gives a $(\theta, \varepsilon, M, c/2)$-$\alpha$-match between $I_t(u)$ and $I_t(v)$. ∎

COROLLARIES. (1) (See B. Weiss' proposition 7.4 in [30].) *Let $\varepsilon > 0$ be given and $c_1 = c(\varepsilon)/2$. Let $c_2 = c_1 + \frac{1}{2}c_1(1 - c_1)$. For any $\theta > 0$ and $M > 0$ there is $n_1 = n_1(\theta, \varepsilon, M)$ s.t. if $t \geqq n_1$ then for any $u, v \in M$ $I_t(u)$ and $I_t(v)$ are $(\theta, \varepsilon, M, c_2)$-$\alpha$-matchable.*

PROOF. Let $\theta_1 < \theta/2$ be so small that $c_1(1 - c_1 - \theta_1) > \frac{1}{2}c_1(1 - c_1)$. Take $n_0 = n_0(\varepsilon, M, \theta_1)$ as in Lemma 3 and let $n_1 = n_0(\varepsilon, n_0, \theta_1)$. One can see that if $t \geqq n_1$ then for any $u, v \in M$, $I_t(u)$ and $I_t(v)$ are $(\theta, \varepsilon, M, c_2)$-$\alpha$-matchable. ∎

(2) *Let $\varepsilon > 0$ be given and $c_{k+1} = c_k + \frac{1}{2}c_k(1 - c_k)$, $c_1 = c(\varepsilon)/2$. Then for any $\theta > 0$, $M > 0$ there is $n_k = n_k(\varepsilon, M, \theta)$ s.t. if $t \geqq n_k$ then for any $u, v \in M$, $I_t(u)$ and $I_t(v)$ are $(\theta, \varepsilon, M, c_k)$-$\alpha$-matchable.*

This follows from (1) by induction on k.

(3) *Since $c_k \to 1$ $k \to \infty$ and θ is arbitrary we get from (2): given $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon)$ s.t. if $t \geqq t_0$ then for any $u, v \in M$, $I_t(u)$ and $I_t(v)$ are ε-α-matchable.*

PROOF OF THEOREM 1. (3) says that h_t satisfies the condition of Lemma 1. By this lemma β is LB for (X, ψ, ν). In the same way we may construct an increasing sequence of partitions $\beta < \beta_1 < \beta_2 < \cdots$ s.t. each β_i is LB and $\vee_i \beta_i$ is the partition of X into points. This implies that ψ is LB (see corollary 4.8 and theorems 6.5, 6.7 in [30]). Since h_t and H_t are uniquely ergodic [6] ψ is a cross-section for both of the flows. This implies that h_t and H_t are LB. ■

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